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TOPICS IN COMPLEX ANALYSIS
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SHEET 7

Solution for 'Topics in complex analysis'

(29/10/2025)

H 7.1 (Holomorphic interpolation)

Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a closed discrete set and $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be an arbitrary countable set. Show that there exists a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(a_n) = b_n$ for all $n \in \mathbb{N}$.
Hint: Combine the Weierstrass product theorem with the Mittag-Leffler theorem for a suitable sequence of principal parts.

Solution H 7.1:

Since the set $\{a_n\}_{n\in\mathbb{N}}$ is closed and discrete in $\mathbb{C}$, we can apply the Weierstrass product theorem to find a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $Z(g) = \{a_n\}_{n\in\mathbb{N}}$ and each zero has multiplicity 1. In particular, $g'(a_n) \neq 0$ for all $n \in \mathbb{N}$. Due to the Mittag-Leffler theorem there exists a holomorphic function $h : \mathbb{C} \setminus \{a_n\}_{n\in\mathbb{N}} \to \mathbb{C}$ such that at each a_n its principal part is given by $q_n(z) = \frac{b_n}{g'(a_n)}(z-a_n)^{-1}$. We define $f : \mathbb{C} \setminus \{a_n\}_{n\in\mathbb{N}} \to \mathbb{C}$ by $f(z) = g(z)h(z)$ and claim that all singularities are removable with $\lim_{z\to a_n} f(z) = b_n$. Then the holomorphic extension satisfies the claimed property. Near a point a_n we can write $g(z) = g'(a_n)(z-a_n) + \tilde{g}(z)(z-a_n)^2$ and $h(z) = q_n(z) + \tilde{h}(z)$ with $\tilde{g}$ and $\tilde{h}$ holomorphic. Hence

$$\lim_{z\to a_n} g(z)h(z) = \lim_{z\to a_n} \Big(g'(a_n)(z-a_n) + \tilde{g}(z)(z-a_n)^2\Big)\Big(\frac{b_n}{g'(a_n)}(z-a_n)^{-1} + \tilde{h}(z)\Big) = b_n.$$

□

H 7.2 (Representation of meromorphic functions as quotients)

Let $h : \mathbb{C} \to \mathbb{C}$ be a meromorphic function (i.e. its singularities are isolated and only poles of finite order). Show that there exist two entire functions $f, g : \mathbb{C} \to \mathbb{C}$ with no common zeros such that $h = \frac{f}{g}$.

Solution H 7.2:

Without loss of generality we may assume that $h \not\equiv 0$. Let us denote by $P(h)$ the set of poles of h. Then $P(h)$ and $Z(h)$ are disjoint closed discrete sets. Indeed, $P(h)$ cannot have accumulation points: if such an accumulation point $z \in \mathbb{C}$ were a pole of h then it would not be isolated, but otherwise h is holomorphic at z, hence also in a neighborhood of z, which contradicts the accumulation of poles at z.
Due to the Weierstrass product theorem we find an entire function $g : \mathbb{C} \to \mathbb{C}$ such that $Z(g) = P(h)$ and $o_z(g)$ is equal to the order of the pole of h at z, for all $z \in P(h)$. Then $g \not\equiv 0$, and setting $f = gh$ it follows that f can be extended to an entire function, since the poles of h are cancelled by the zeros of g (i.e. f is bounded near each point $z \in P(g)$). By construction we have $Z(f) \cap Z(g) = \emptyset$. □

H 7.3 (Weierstrass product theorem on open sets)
Let $U \subset \mathbb{C}$ be an open set and let $\{a_n\}_{n\in\mathbb{N}} \subset U$ be a sequence with no accumulation points in U. Set $o_n := \#\{k \in \mathbb{N} : a_k = a_n\}$. We claim that there exists a holomorphic function $f : U \to \mathbb{C}$ such that $Z(f) = \{a_n\}_{n\in\mathbb{N}}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbb{N}$. Moreover, as we shall see in the proof, the function f can be taken as an infinite product.
The argument splits into two steps. Similarly to the Mittag-Leffler theorem, we recenter some of the Weierstrass factors and replace $E_n\big(\frac{z}{a_n}\big)$ by $E_n\big(\frac{a_n-c_n}{z-c_n}\big)$, for a suitable sequence c_n. We denote by S' the set of accumulation points (in $\mathbb{C}$) of the sequence $\{a_n\}_{n\in\mathbb{N}}$. If $S' = \emptyset$ there is nothing to prove as we can apply Theorem 4.2. Hence assume that $S' \neq \emptyset$.

a) Suppose that there exists a sequence $\{c_n\}_{n\in\mathbb{N}} \subset S'$ such that $\lim\limits_{n\to+\infty} |a_n - c_n| = 0$. Show that the infinite product

$$f(z) = \prod_{n=1}^{\infty} E_n\Big(\frac{a_n - c_n}{z - c_n}\Big)$$

converges locally normally on U and satisfies $Z(f) = \{a_n\}_{n\in\mathbb{N}}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbb{N}$.

b) Split the set $S := \{a_n\}_{n\in\mathbb{N}}$ as in Lemma 2.7 and use Lemma 2.8 to conclude the proof by combining a) and Theorem 4.2.

Solution H 7.3:
a) Let $K \subset U$ be compact. Then there exists $\varepsilon > 0$ such that $\operatorname{dist}(K, \partial U) \geq \varepsilon > 0$, since the distance is attained (as $\partial U = \overline{U} \setminus U$ is closed). From $c_n \in S' \subset \partial U$ we deduce that that there exists $n(K) \in \mathbb{N}$ such that for all $n \geq n(K)$ we have

$$K \subset \{z \in U : \underbrace{|z - c_n|}_{\geq \varepsilon \text{ on } K} \geq 2\underbrace{|a_n - c_n|}_{\to 0}\}.$$

Hence for $n \geq n(K)$ it holds that $\Big|\frac{a_n-c_n}{z-c_n}\Big| \leq \frac{1}{2}$, so that from Lemma 4.1 we infer that

$$\sum_{n\geq n(K)} \sup_{z\in K} \Big|E_n\Big(\frac{a_n - c_n}{z - c_n}\Big) - 1\Big| \leq \sum_{n\geq n(K)} 2^{-(n+1)} < +\infty.$$

Thus the infinite product f converges locally normally on U. Since each E_n has a simple zero at $z = 1$ and $a_n \neq c_n$ for all $n \in \mathbb{N}$ (recall that $a_n \in U$ and $c_n \notin U$) we deduce from Lemma 3.11 that $Z(f) = \{a_n\}_{n\in\mathbb{N}}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbb{N}$.
b) Splitting $S = S_1 \sqcup S_2$ as in Lemma 2.7 we obtain that $S_1 = \{a_{n,1}\}_n$ is closed and therefore $S' = S_2'$. Due to Lemma 2.8, $S_2 = \{a_{n,2}\}_{n\in\mathbb{N}}$ has no accumulation points in U and there exists a sequence $\{c_n\}_{n\in\mathbb{N}} \subset S_2' = S'$ such that $\lim_{n\to+\infty} |c_n - a_{n,2}| = 0$. Since S_1 is closed in $\mathbb{C}$, we can apply Theorem 4.2 to conclude that the entire function $f_1 : \mathbb{C} \to \mathbb{C}$ defined by

$$f_1(z) = \prod_n E_n\big(\tfrac{z}{a_{n,1}}\big)$$

has zeros exactly in $\{a_{n,1}\}_n$ and with the correct multiplicity (here we also used that $0 \notin S_1$). On S_2 we can apply a) to deduce that the holomorphic function $f_2 : U \to \mathbb{C}$ given by

$$f_2(z) = \prod_{n=1}^{\infty} E_n\Big(\frac{a_{n,2} - c_n}{z - c_n}\Big)$$

has zeros exactly in $\{a_{n,2}\}_{n\in\mathbb{N}}$ and with the correct multiplicity. Since S_1 and S_2 are disjoint, the function $f = f_1 \cdot f_2$ satisfies the desired properties. □

H 7.4 (Blaschke condition)
Prove that if $f : B_1(0) \to \mathbb{C}$ is holomorphic, bounded, and not identically zero, and $z_1, z_2, \ldots$ are its zeros (with $|z_n| < 1$ and listed with multiplicity), then

$$\sum_n (1 - |z_n|) < \infty.$$

Hint: Use Jensen's formula.

Solution H 7.4:
We may assume that $f(0) \neq 0$, since the $z_n = 0$ do not affect convergence of the series.
Let $0 < r < 1$ and apply Jensen's formula to f with radius r such that f has no zeros in $\partial B_r(0)$. Then

$$\log|f(0)| = \sum_{|z_n|<r} \log\left(\frac{|z_n|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|\,dt.$$

The integral term is bounded by a constant $M > 0$ (independent of r), since f is bounded. Hence

$$\sum_{|z_n|<r} \log\left(\frac{r}{|z_n|}\right) \leq M - \log|f(0)|.$$

The function $r \mapsto \log\left(\frac{r}{|z_n|}\right)$ is increasing for $r > |z_n|$, and as $r \to 1^-$ it increases to $-\log|z_n|$. Therefore, by monotone convergence (taking a limit along radii $r \neq |z_n|$ for all $n \in \mathbb{N}$) we have

$$-\sum_n \log|z_n| = \lim_{r\to 1^-} \sum_{|z_n|<r} \log\left(\frac{r}{|z_n|}\right) < \infty.$$

Thus the desired result follows from the inequality $x \leq -\log(1-x)$ for $0 < x < 1$, itself a consequence of the series expansion $-\log(1-x) = \sum_{k=1}^{\infty} x^k/k$.

□

H 7.5 (Blaschke products)
In this problem, we discuss Blaschke products, which are bounded analogues in the disc of the Weierstrass products for entire functions.

a) Show that for any $0 < |\alpha| < 1$ and $|z| \leq r < 1$, we have the inequalities

$$\left|\frac{\alpha + |\alpha| z}{(1-\overline{\alpha}z)\alpha}\right| \leq \frac{1+r}{1-r} \qquad \text{and} \qquad \left|\frac{\alpha - z}{1-\overline{\alpha}z}\right| \leq 1.$$

b) Let $\{\alpha_n\}_n$ be a sequence in $B_1(0)$ such that $\alpha_n \neq 0$ for all n and

$$\sum_n (1 - |\alpha_n|) < \infty.$$

Note that this will be the case if $\{\alpha_n\}_n$ are the zeros of a bounded holomorphic function on the unit disc (see Exercise H 7.4). Show that the product

$$f(z) = \prod_n \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n}$$

converges uniformly for $|z| \leq r < 1$, hence defines a holomorphic function on the unit disc having precisely the zeros α_n and no other zeros. Show also that $|f(z)| \leq 1$.

Solution H 7.5:
a) Observe that

$$\frac{\alpha + |\alpha| z}{(1-\overline{\alpha}z)\alpha} = 1 + \frac{z|\alpha|(|\alpha|+1)}{\alpha(1-\overline{\alpha}z)}.$$

Since $|z| \leq r$ and $|\alpha| < 1$ we have $\frac{|\alpha|+1}{|1-\overline{\alpha}z|} \leq \frac{2}{1-|\alpha z|} \leq \frac{2}{1-r}$, so

$$\left|\frac{\alpha + |\alpha| z}{(1-\overline{\alpha}z)\alpha}\right| \leq 1 + \frac{2r}{1-r} = \frac{1+r}{1-r}.$$

The second inequality follows from

$$|\alpha - z|^2 \leq |1 - \overline{\alpha}z|^2 \iff |\alpha|^2 + |z|^2 \leq 1 + |\alpha z|^2 \iff 0 \leq (1-|\alpha|^2)(1-|z|^2).$$

b) We prove the local normal convergence of the product for $f(z)$, which implies that it is holomorphic. Write the factors $1 + g_n(z)$ of the product defining $f(z)$ as

$$1 + g_n(z) := \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n} = 1 + \frac{(|\alpha_n| - 1)(\alpha_n + |\alpha_n| z)}{(1 - \overline{\alpha_n} z)\alpha_n}.$$

Since the compact sets $\overline{B_r(0)}$ for $0 < r < 1$ cover $B_1(0)$, the local normal convergence follows from item a) as

$$\sup_{|z| \leq r} \sum_n |g_n(z)| \leq \sup_{|z| \leq r} \sum_n (1 - |\alpha_n|) \left|\frac{\alpha_n + |\alpha_n| z}{(1 - \overline{\alpha_n} z)\alpha_n}\right| \leq \frac{1+r}{1-r} \sum_n (1 - |\alpha_n|) < \infty.$$

It remains to prove that $|f(z)| \leq 1$, which follows since each factor satisfies $|1 + g_n(z)| \leq 1$, due to the second inequality of item a).

$\square$